Elasticity from the Force Network Ensemble in Granular Media

Srdjan Ostojic and Debabrata Panja

Institute for Theoretical Physics, Universiteit van Amsterdam,
Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands
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Transmission of forces in static granular materials are studied within the framework of the force network ensemble, by numerically evaluating the mechanical response of hexagonal packings of frictionless grains and rectangular packings of frictional grains. In both cases, close to the point of application of the overload, the response is non-linear and displays two peaks, while at larger length-scales it is linear and elastic-like. The cross-over between these two behaviors occurs at a depth that increases with the magnitude of the overload, and decreases with increasing friction.

Mechanical properties of granular materials, or assemblies of static macroscopic particles have attracted much attention in recent years [1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18]. Under external stresses, granular materials present a solid-like behavior, which is due to an intricate network of repulsive forces between particles in contact. The discrete nature and high spatial inhomogeneity of this force network are the source of many remarkable phenomenological properties [1, 2], which have raised doubts about the relevance of a continuum, elastic mechanical description traditionally used in the engineering community [19]. As a result, a number of alternative descriptions have been proposed for the propagation of stresses [3, 4].

To distinguish between the various theoretical approaches, experiments [5, 6, 7, 8, 9] and numerics [10, 11, 12, 13] examined the mechanical response of granular packings [2, 20], i.e. the variation of forces due to a small, localized overload. The experiments identified a strong influence of the underlying geometry of grains. In a disordered packing of polydisperse grains, the maximal response was found vertically below the point of application of the overload, with the width of the response function linearly increasing with depth, as predicted by isotropic elasticity theory. In contrast, in a crystalline arrangement of monodisperse grains, the response is maximal along a set of preferred directions, resembling the predictions of anisotropic elasticity [14]. These two aspects were reconciled by molecular dynamics simulations [10] suggesting that the anisotropic response in ordered packings is only a short-range, non-linear effect, which gives way to isotropic elastic behavior above a cross-over length scale.

Altogether, experimental and numerical results thus unambiguously indicate the elastic nature of stress propagation in granular materials. It however remains unclear how such a picture emerges from the mesoscopic physics, and in particular from the underlying discrete force network. Many aspects of this highly disordered structure have been successfully described using the so-called *force network ensemble* [15]. This statistical approach ignores
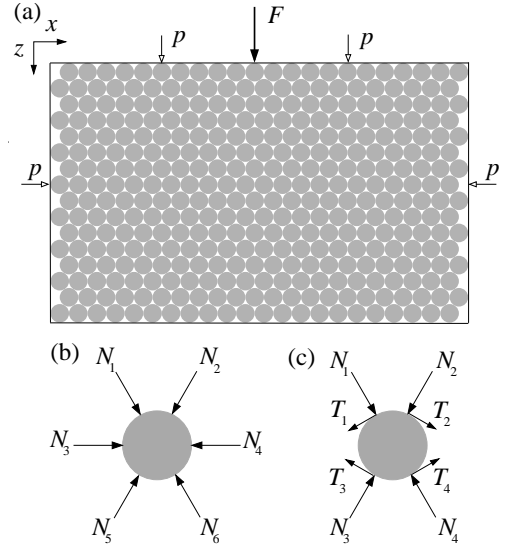

FIG. 1: The geometries studied: (a) ordered packing of L layers of grains confined by a pressure p on the boundaries; each layer contains $2L + 1$ grains; an overload F is applied on the grain x_0 in the top layer. (b) in the frictionless case, each grain interacts via normal forces with six neighbors; (c) in the frictional case, each force has both a normal and a tangential components, but there are no contacts between grains on the same layer, and the horizontal boundary conditions are chosen to be periodic.

the details of the contact mechanics, and takes into account only the fundamental constraint that the forces must balance on each grain. For a given geometrical arrangement, this requirement does not specify a unique force network, but a whole ensemble. The central idea behind the force ensemble approach is that generic features of forces might be described by the statistics of this ensemble. Remarkably, this approach was found to account for a number of properties of force networks, in particular the distribution of force magnitudes [15, 16], the geometrical patterns of large forces [17], and the anisotropy and yielding of force networks under shear [18].

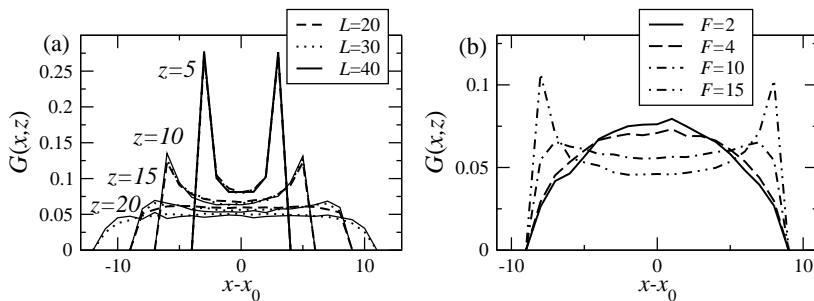In this Letter, we show that the force network ensem-

FIG. 2: Simulation results for the mean response $G(x, z)$ in the frictionless case: (a) $G(x, z)$ for $F = 10$ as function of $x - x_0$, at four different depths z, and for three different system sizes L; (b) $G(x, z)$ at the depth $z = 15$, for four different values of the overload F (in units of p).

ble describes the transmission of forces at an unexpected level of detail. By numerically studying the mechanical response within this framework, we find a long length-scale elastic behavior as well as a short length-scale non-linearity. We examine two distinct two-dimensional settings: (i) a hexagonal packing of frictionless grains and (ii) a rectangular packing of frictional grains, both confined by an isotropic pressure. In both cases, close to the point of application of the overload the response is double-peaked and non-linear; but deeper into the packing an increasing amount of force is transmitted towards the center of the packing. At a given depth, a cross-over to a single-peak occurs. Below that depth, the response is essentially linear and elastic-like. The cross-over depth increases with increasing overload, but, for frictional grains, decreases with friction.

Mechanical response from the force ensemble— In the spirit of Edward's statistical approach [3, 21], the force network ensemble ignores the details of the dynamical history of the granular packing, and considers the ensemble of all possible force networks for a given contact geometry. This ensemble is restricted by the constraints that the forces must balance on each grain, and be consistent with imposed boundary stresses. In generic granular packings, the number of equations of mechanical balance is smaller than the number of unknowns [22]. The ensemble $\mathcal{E}$ of allowed force networks is thus a high-dimensional convex set [23, 24], whose boundaries are delimited by the requirement that all normal forces are repulsive, and that all tangential forces satisfy Coulomb's condition. By analogy with the microcanonical ensemble, every point of $\mathcal{E}$ is considered equally likely.

The shape of $\mathcal{E}$ intrinsically depends on the boundary forces imposed on the packing. If we call $\mathcal{E}_p$ the ensemble corresponding to a uniform pressure p on the boundaries and $\mathcal{E}_{p+F}$ the ensemble corresponding to pressure p with and additional vertical overload F on a given grain of the boundary, the average mechanical response can be defined as $G(i, j) = (\langle W_{ij} \rangle_{\mathcal{E}_{p+F}} - \langle W_{ij} \rangle_{\mathcal{E}_p})/F$, where W_{ij} is the total vertical force on the grain (i, j), and the

brackets denote averages over respective ensembles. Note that in the spirit of Edwards' approach, this definition ignores the dynamics generated by the application of the overload, and considers only the possible outcomes. In particular, there is no assumption of linearity.

To compute $G(i, j)$, we sample the force ensemble using Monte Carlo simulations. We first identify a parametrization of $\mathcal{E}$ [26]. Starting from an interior point, we then implement a random walk which is not allowed to leave $\mathcal{E}$ [25]. Such a procedure satisfies a detailed balance which leads to a uniform sampling.

Hexagonal packing of frictionless grains— Consider a hexagonal packing of massless and frictionless disks confined by an external pressure applied on the walls of the packing (c.f. Fig. 1). In Refs. [26, 27], we examined this packing under zero vertical and large horizontal pressure. While this situation allows an instructive comparison between the force ensemble and the q-model [28], physically it corresponds to a singular limit, where there is no relevant scale for the applied overload [30]. Here we study the practically relevant case where the pressure is isotropic. We have obtained similar results (not shown here) for grains confined by gravity instead of vertical pressure.

In the absence of friction, the contact forces are normal to the grains. As each grain shares six contacts with its neighbors [c.f. Fig. 1 (b)], there are three independent forces and two equations of force balance per grain. The force ensemble can thus be parametrized by specifying one force per grain, for example N_4. In this case, the dimension of $\mathcal{E}$ is $L(2L + 1)$, and its boundaris are determined by the requirement that all the forces in the packing be repulsive. The sampling was implementing via a "hit and run" algorithm [25].

The results obtained from Monte Carlo simulations are shown in Fig. 2. Fig. 2(a) shows the response at various depths to an overload $F = 10$ (in units of pressure), for several system sizes. Close to the top of the packing, $G(i, j)$ displays two sharp peaks along the lattice directions emanating from the point of application of the overload. Deeper in the packing, the two peaks broaden, and
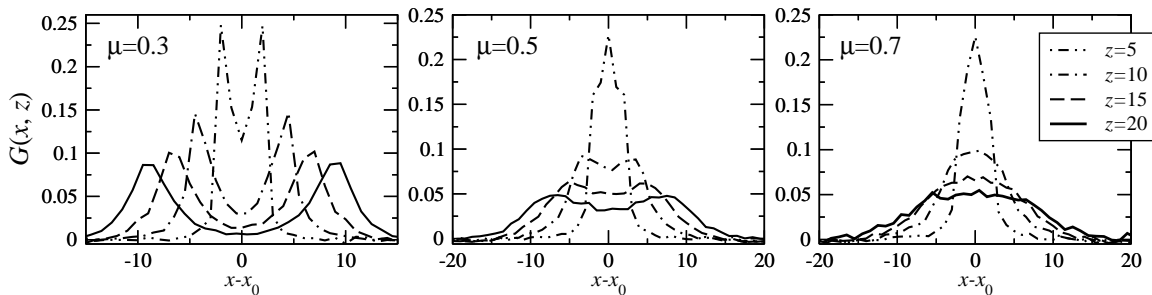
FIG. 3: Simulation results for the mean response $G(x, z)$ at different depths z in the frictional case. The value of the applied force is $F = 3$ (in units of p). Each graph corresponds to a different value of the coefficient of friction μ. The total number of layers is $L = 20$.

an increasing amount of load is transfered towards the center. Progressively the response becomes flatter, and at a given depth the two peaks disappear completely. Below that depth, only a broad central peak persists, and its width increases linearly with depth. These results are independent of the size of the simulated system.

The dependence on the magnitude of the overload F is displayed in Fig. 2(b), where the response is shown at a depth $i = 15$. For small overloads, $G(i, j)$ is single peaked, and essentially independent of F, which shows that the variations of the forces are linear in applied overload. As F is increased, the response becomes flatter and progressively two peaks appear. For large overloads, these peaks are located on the lattice directions. In conclusion, for increasing overloads, the crossover between a double-peaked and a single-peaked response is shifted deeper in the packing.

Rectangular packing of frictional grains— In the presence of friction, each contact force possesses a tangential component in addition to the normal one. Moreover, torque balance on each grain must be taken into account. This implies that there are three unknowns per grain in a hexagonal packing, a situation for which we found reasonably-sized systems impossible to simulate without the recourse of impractically long computer runtimes. We therefore considered a rectangular packing, in which grains in the same layer are not in contact (c.f. Fig. 1 (c)), a geometry previously studied in [29], but in a different framework [31]. In the absence of friction, the response of a rectangular packing is trivial, as the overload propagates exclusively along the lattice directions. This setting thus allows to isolate the effects of friction, which is the *only* source of disorder.

On each grain, there are now four unknown forces and three equations, so that the force ensemble can again be parametrized by specifying one force per grain, which we choose to be T_4. The boundaries of $\mathcal{E}$, whose dimension is again $L(2L + 1)$, are determined by the requirement that all normal forces be repulsive and that all tangential forces satisfy Coulomb's criterion $|T_i| \le \mu N_i$, where μ is

the coefficient of friction. Due to this second requirement, the shape of $\mathcal{E}$ is complicated, and we use a "ball-walk" sampling [25].

The results of the Monte Carlo simulations, with F expressed in units of p are displayed in Figs. 3 and 4. In Fig. 3, the response to an overload $F = 3$ is shown at different depths, for different coefficients of friction μ. For small μ, the response consists of two peaks along the lattice directions. For $\mu = 0.3$, these peaks broaden essentially linearly with depth (cf. Fig. 5), in agreement with experimental findings [7]. For $\mu = 0.5$, the peaks are very broad, and merge into a single one below a given depth. For $\mu = 0.7$, the response consists exclusively of a single, central peak, whose width broadens linearly with depth (cf. Fig. 5). Altogether, this results indicate that the depth of the crossover between double and single-peaked response decreases with increasing μ.

The dependence of the response on the magnitude of the overload F is displayed in Fig. 4 for different coefficients of friction μ. Clearly, as F is increased, the two peaks become more prominent, and propagate deeper in the packing, as in the frictionless case. For small F, in the single-peaked regime, the response is independent of the overload, in other words it is linear.

Discussion— The force ensemble yields remarkably accurate predictions for the mechanical response of ordered packings. In particular, our results for the dependence of the crossover depth on applied overload and friction are in all points comparable to those of molecular dynamics simulations [10], which describe the grains at a much greater level of detail. The force ensemble thus provides a simple theoretical framework which predicts both the linear response described by elasticity theory, and the experimentally relevant non-linear behavior.

A closer look at the horizontal forces on the grains indicates that the reason for this cross-over can be intuitively understood by comparing the typical magnitudes of W_{ij} with the typical size δ of fluctuations in the horizontal forces on the grains. In our formulation, these fluctuations are parametrized by $N_4 - N_3$ in the frictionless
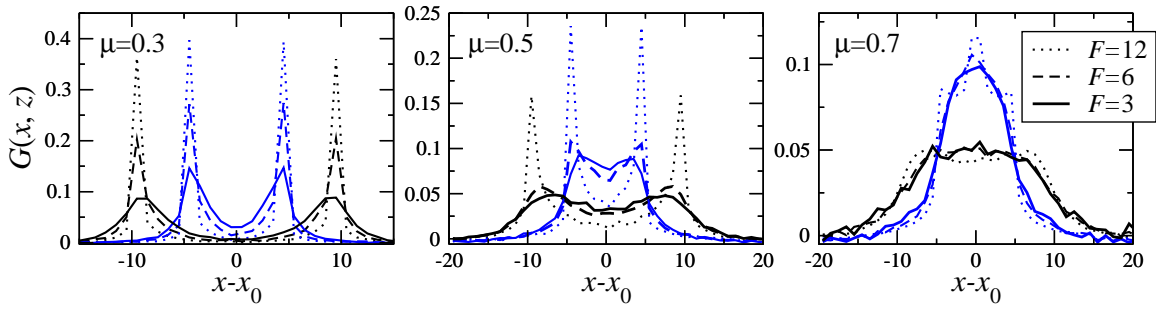
FIG. 4: The mean response $G(x,z)$ for different values of the overload F (in units of p) for the frictional case. Data are shown for two different layers: $z = 20$ (black), and $z = 10$ (blue). Each graph corresponds to a different value of μ. The total number of layers is $L = 20$.
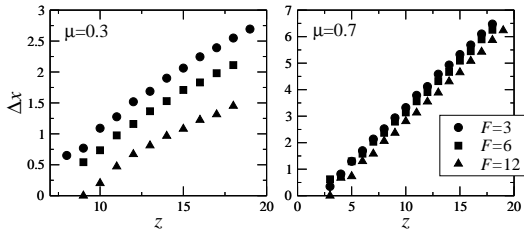

FIG. 5: Broadening of the response, quantified by $\Delta x(z) = \left(\int x^2 G(x,z) dx - \left[\int x G(x,z) dx \right]^2 \right)^{1/2}$. For $\mu = 0.3$, the sum was restricted to $x > x_0$.

case, and T_4 in the frictional case. Their sizes are determined by the requirement that normal forces be positive and that tangential ones obey Coulomb's criterion, so that δ increases with p and μ. For $W_{ij} \gg \delta$, the packing effectively becomes rectangular (although horizontal contacts are not broken), and the overload propagates mainly along the lattice directions. The scenario is indeed confirmed by Fig. 2(b) — increasing F/p implies increasing W_{ij}/p, which in turn makes the peaks in the response function penetrate deeper into the packing. To complement this observation, we also find that the crossover occurs at a layer i where $W_{ij} \sim \delta_{ij}$.

We believe that it is possible to use our findings in the above paragraphs to obtain a field-theory type formulation for the mechanical response of hard grain packings, and it remains an important future direction of our work.

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[30] As a consequence, the response is independant of the overload, and self-similar with the system size.
[31] Our results differ significantly from those of Ref. [29], wherein T_4 was chosen independently on each grain, in the spirit of the q-model [28]. In Refs. [26, 27], we showed that the stochastic prescription of the q-model on a hexagonal packing differs significantly from that of the force network ensemble. It is thus not a surprise to

find a distinction between the two also in a rectangular
packing of frictional grains.